Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis -I

Final Examination | Date : 5 December 2022
Maximum marks: 100 | Time: 3 hours
Instructor: B V Rajarama Bhat

Notation: In the following when interval $[a, b]$ is considered it is assumed that $a, b \in \mathbb{R}$ and $a < b$.

(1) (i) Write down a bijection $h : (0, 1) \to [0, 1]$.
(ii) Show that there is no continuous bijection $g : (0, 1) \to [0, 1]$. [15]

(2) Let X, Y be bounded subsets of $[0, \infty)$. Take $Z := \{xy : x \in X, y \in Y\}$.
(i) Show that
$$\sup(Z) = \sup(X). \sup(Y).$$
(ii) Show that the same result may not hold if A, B are general bounded subsets of $\mathbb{R}$. [15]

(3) Let $u : [a, b] \to \mathbb{R}$ be a continuous function. Define $v : [a, b] \to \mathbb{R}$ by
$$v(x) = \begin{cases} 4 & \text{if } u(x) \leq 4; \\ u(x) & \text{if } u(x) \geq 4. \end{cases}$$
Show that v is continuous. [15]

(4) Let $f : [a, b] \to \mathbb{R}$ be a function. Show that f is differentiable at a if and only if for every **decreasing** sequence $\{x_n\}_{n\geq 1}$ in (a, b), converging to a,
$$\lim_{n\to\infty} \frac{f(x_n) - f(a)}{x_n - a}$$
exists. [15]

(5) Let $m : [0, 1] \to \mathbb{R}$ be a differentiable function. Suppose the set of fixed points of m:
$$F := \{x \in [0, 1] : m(x) = x\}$$
is infinite. Show that there exists $y \in F$ such that $m'(y) = 1$. [15]

(6) Compute points of local minima and local maxima for the function $k : [1, 4] \to \mathbb{R}$ defined by
$$k(x) = \begin{cases} 2x & \text{if } 1 \leq x < 2; \\ 2 + (x - 3)^2 & \text{if } 2 \leq x \leq 4. \end{cases}$$

(7) Let α, β be two real numbers with $0 < \alpha < \beta < 1$. Define
$$b_n = \begin{cases} (-1)^n\alpha^n & \text{if } n \text{ is odd}; \\ (-1)^n\beta^n & \text{if } n \text{ is even}. \end{cases}$$
(i) Determine as to whether $\sum_{n=1}^{\infty} b_n$ is absolutely convergent or not.
(ii) Compute $\sum_{n=1}^{\infty} b_n$. [15]